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# From quantum universal enveloping algebras to quantum algebras

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**Abstract**
The 'local' structure of a quantum group $G_q$ is currently considered to be an infinite-dimensional object: the corresponding quantum universal enveloping algebra $U_q(g)$, which is a Hopf algebra deformation of the universal enveloping algebra of an $n$-dimensional Lie algebra $g = Lie(G)$. However, we show how, by starting from the generators of the underlying Lie bialgebra $(g, \delta)$, the analyticity in the deformation parameter(s) allows us to determine in a unique way a set of $n$ 'almost primitive' basic objects in $U_q(g)$, which could be properly called the 'quantum algebra generators'. So, the analytical prolongation $(g_q, \Delta)$ of the Lie bialgebra $(g, \delta)$ is proposed as the appropriate local structure of $G_q$. Besides, as in this way $(g, \delta)$ and $U_q(g)$ are shown to be in one-to-one correspondence, the classification of quantum groups is reduced to the classification of Lie bialgebras. The $su_q(2)$ and $su_q(3)$ cases are explicitly elaborated.

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## 1. Introduction

Quantum groups are a non-commutative generalization of Lie groups endowed with a Hopf algebra structure [1, 2]. Some attempts to get structural properties of these objects have been previously considered (see [3, 4] for a prescription to get the quantum coproduct—but not the deformed commutation rules—for a wide class of examples). Moreover, to our knowledge, a general investigation concerning the uniqueness of this quantization process has not yet been given and only restrictive results for certain deformations of simple Lie algebras have been obtained (see [5], chapter 11). As a consequence of the above-mentioned facts, a complete

classification of quantum groups in the spirit of Cartan cannot be found in the literature (see [6, 7] and references therein).

We would like to stress that both in Lie group theory and in their physical applications, the infinitesimal counterpart of a Lie group transformation—i.e. its Lie algebra $g$—plays a fundamental role, since it is interpreted as the local (around the identity) symmetry. Correspondingly, the local counterpart of quantum groups was soon algebraically identified through Hopf algebra duality, presents a wealth of interesting mathematical properties, and has been also applied in different physical contexts.

However, such an infinitesimal counterpart of a quantum group $G_q$ is not a deformed Lie algebra, but a quantum universal enveloping algebra $U_q(g)$: a Hopf algebra deformation of the universal enveloping algebra $U(g)$ of $g$, i.e., a deformation of the infinite-dimensional object that has as a basis the set of all ordered monomials of powers of the generators of $g$. This means that quantum deformations lead us locally (therefore, geometrically) to structures which are quite different from Lie algebras. In particular, when we consider a deformation of an $n$-dimensional Lie algebra, only the infinite-dimensional algebra $U_q(g)$ makes sense despite that its Poincaré–Birkhoff–Witt (PBW) basis is constructed in terms of a basic set of elements of the same dimension of the Lie algebra. Indeed, contrarily to the non-deformed case, where inside all sets of basic elements the vector space of generators is univocally defined, in $U_q(g)$ there are an infinitude of basic sets coexisting on the same footing.

This problem of the basis underlies many difficulties encountered when a precise physical/geometrical meaning has to be assigned to the $U_q(g)$, as in the context of quantum deformations of spacetime symmetries. In that case, it is well known that the models so obtained depend on the choice of different bases (for instance, the bicrossproduct one [8]), and that different possibilities are related through nonlinear transformations. The aim of this paper is to solve this problem providing a universal and computational prescription for the characterization and construction of the $n$-dimensional quantum analogue of a Lie algebra.

To do this, we have to analyse the role and properties of the Lie algebra generators within $U(g)$. Among the infinite possible PBW bases, all of them related by nonlinear invertible transformations, the generators determine the only basis closed under linear commutation rules and whose tensor product representations are constructed additively. The latter property can be stated in Hopf-algebraic terms as the Friedrichs theorem [9]: the only primitive elements $\{X_j\}$ in $U(g)$ (i.e. the elements such that $\Delta(X_j) = \Delta_{(0)}(X_j) := 1 \otimes X_j + X_j \otimes 1$) are just the generators of $g$ as a Lie algebra. In this way, the generators of $g$ become distinguished elements of $U(g)$.

The additivity of generators in representation theory is the reason why in physical applications we are used to disregarding $U(g)$ as a mathematical curiosity and focusing on the quite more manageable Lie algebra. However, we realize immediately that the situation changes drastically in $U_q(g)$, where the law for the construction of tensor product representations (coproduct) includes nonlinear functions and no primitive bases exist. In this paper we show that, among the infinitely many possible bases, there is only one (that we will call the 'almost primitive' basis) where the coproducts are 'as primitive as possible', since all inessential terms have been removed. Indeed, the only changes from the primitive coproducts are those imposed by the bialgebra cocommutator $\delta$ to be consistent with the Hopf algebra postulates. This almost primitive basis is proposed as the true deformation of the Lie algebra and is thus called *quantum algebra*.

It is well known that Lie group theory is based on analyticity with respect to group parameters. In the same way, analyticity in the deformation parameter(s) will give us the keystone for the identification of the proper quantum algebra, which will be defined as the *n-dimensional vector space* $(g_q, \Delta) \subset U_q(g)$ *obtained as analytical prolongation of the Lie*

*bialgebra* $(g, \delta)$. (Note that analyticity in the deformation has already played a useful role in quantum algebras, for instance in their contractions [10].)

In this analytical prolongation, the Lie bialgebra cocommutator map $\delta$ describes the first-order deformation and can be considered as the derivative at the origin of the quantum coproduct. This $\delta$, together with the zero-order deformation (the Lie-Hopf algebra) and the coassociativity of the coproduct will allow us to construct order-by-order the deformed coproduct. The commutators (not $q$-commutators) are then obtained by imposing the homomorphism property for the coproduct.

Summarizing, in this paper we are attempting to describe the 'commutative' (in a broad sense) diagram

$$\begin{array}{ccc} (g, \Delta_{(0)}) & \xleftarrow{\text{Friedrichs theorem}} & U(g) \\ \downarrow q & & \downarrow q \\ (g_q, \Delta) & \xleftarrow{\text{Generalized Friedrichs th.}} & U_q(g) \end{array}$$

where a new object, the quantum algebra $(g_q, \Delta)$, is introduced, and its connections with its neighbours in the diagram fully discussed. The vertical lines of the diagram represent the quantization procedure, and the horizontal ones are related to the definition of the basic set of the universal enveloping algebras and their quantum analogues. Remember that, for a given Lie–Hopf algebra $(g, \Delta_{(0)})$, several Lie bialgebras $(g, \delta)$ exist and each of them determines one different quantization and, as a consequence, a different diagram.

The paper is organized as follows. In section 2 we describe the analytical approach to the problem and, in particular, the relation between the Lie algebra $(g, \Delta_{(0)})$ and its analytical prolongation in the direction of $\delta$, the quantum algebra $(g_q, \Delta)$. Moreover, in order to make the approach more clear, we present in section 3 the construction of the standard deformation of $su(2)$. Section 4 is a true application as it exhibits the standard quantization of $su(3)$ with all generators, commutation relations and coproducts. Section 5 is devoted to revisiting the first horizontal line of the diagram, i.e. the one-to-one connection between $U(g)$ and $(g, \Delta_{(0)})$ (Friedrichs theorem), in such a way that can be generalized to connect—always in a one-to-one way—$U_q(g)$ and $(g_q, \Delta)$, a subject that is discussed in section 6. Finally, some conclusions close the paper.

## 2. Analytical quantization: $(g, \Delta_{(0)}) \to (g_q, \Delta)$

As it is well known, the quantum universal enveloping algebra $U_q(g)$ is a Hopf algebra that depends on one deformation parameter $z = \log q$ (the generalization to multiparametric deformations is straightforward) and such that, in the limit $z \to 0$ (or $q \to 1$), $U_q(g)$ becomes $U(g)$ and all possible sets of basic elements reduce to a basis in $g$. Also, $U_q(g)$ is the quantization of a given Lie bialgebra $(g, \delta)$ where $g$ is a Lie algebra of dimension $n$, and $\delta : g \to g \otimes g$ is a compatible skew-symmetric map [1]. In particular, $U_q(g)$ is a Hopf algebra such that

$$\delta = \lim_{z \to 0} \frac{\Delta - \sigma \circ \Delta}{2z},$$

$\sigma$ being the flip operator (i.e., $\sigma(A \otimes B) = B \otimes A$). So, $\delta$ can be interpreted as the derivative at the origin of the quantization and $U_q(g)$ is sometimes called a 'quantization of $U(g)$ in the direction of $\delta$'. Such a quantization is usually constructed starting from any PBW basis in $U(g)$. Thus, a univocal correspondence between $U_q(g)$ and $(g, \delta)$ is found, while no general results concerning the uniqueness of the quantization process are known.

Here we present a different quantization procedure:

(1) By using analyticity and coassociativity we find order-by-order the changes induced in $\Delta_{(0)}$ by $\delta$ and we determine in this way the full quantum coproduct $\Delta$.
(2) By using analyticity and the homomorphism property of $\Delta$, we obtain the commutation rules for $g_q$ starting from the known ones of $g$. Thus, the $n$-dimensional $(g_q, \Delta)$ is constructed.
(3) A PBW basis in $U_q(g)$ is built from $g_q$.

We thus construct a unique correspondence $(g, \delta) \to (g_q, \Delta) \to U_q(g)$. Since $(g, \delta)$ is the limit of $(g_q, \Delta)$ and, as shown in section 6, $g_q$ is the only almost primitive basis in $U_q(g)$ (exactly like $g$ is the only primitive basis in $U(g)$), a one-to-one correspondence is found between $(g, \delta)$ and $U_q(g)$. So equivalences in $U_q(g)$ imply equivalences in the Lie bialgebras, and the classification of $U_q(g)$ is carried to the quite simpler classification of Lie bialgebras.

The two main assumptions of the analytical quantization procedure are:

(1) The commutation relations of any basic set $\{Y_j\}(j = 1, 2, \ldots, n)$ of $U_q(g)$ (as well as of $U(g)$) are analytical functions of the $Y_j$.
(2) The quantum coproduct $\Delta$ of $Y_j$ can be written as a formal series

$$\Delta(Y_i) = \sum_{k=0}^{\infty} \Delta_{(k)}(Y_i) = \Delta_{(0)}(Y_i) + \Delta_{(1)}(Y_i) + \cdots \tag{2.1}$$

with $\Delta_{(k)}(Y_i)$ a homogeneous polynomial of degree $k + 1$ in $1 \otimes Y_j$ and $Y_j \otimes 1$.

Since we deal with a Hopf algebra, $\Delta$ has to verify the coassociativity condition

$$(\Delta \otimes 1 - 1 \otimes \Delta) \circ \Delta(Y_i) = 0 \tag{2.2}$$

as well as the homomorphism property

$$\Delta([Y_i, Y_j]) = [\Delta(Y_i), \Delta(Y_j)]. \tag{2.3}$$

Taking into account equation (2.1), equations (2.2) and (2.3) can be rewritten as

$$\sum_{j=0}^{k} (\Delta_{(j)} \otimes 1 - 1 \otimes \Delta_{(j)}) \circ \Delta_{(k-j)}(Y_i) = 0, \qquad \forall k, \tag{2.4}$$

$$\Delta_{(k)}([Y_i, Y_j]) = \sum_{l=0}^{k} [\Delta_{(l)}(Y_i), \Delta_{(k-l)}(Y_j)], \qquad \forall k. \tag{2.5}$$

Note that commutation rules and coproducts in $U_q(g)$ are fully defined from commutation rules and coproducts in $\{Y_j\}$.

In order to deform $(g, \Delta_{(0)}) \to (g_q, \Delta)$ in the direction of $\delta$ we have to introduce the modifications to $\Delta_{(0)}$ imposed by the Lie bialgebra $(g, \delta)$ to be consistent with the coassociativity. Thus we define

$$\Delta_{(1)}(X_i) := z\delta(X_i),$$

by putting to zero arbitrary cocommutative contributions to $\Delta_{(1)}(X_i)$ because they are unrelated to $\delta$ (see sections 5 and 6). Then we can write

$$\Delta(X_i) = \Delta_{(0)}(X_i) + z\delta(X_i) + \mathcal{O}_{(2)}(X_i), \tag{2.6}$$

where $\mathcal{O}_{(m)}(X_i)$ is a series of degree greater than $m$ in $X_j \otimes 1$ and $1 \otimes X_j$. Because of (2.1), $\mathcal{O}_{(2)}(X_i)$ can be written as

$$\mathcal{O}_{(2)}(X_i) = \Delta_{(2)}(X_i) + \mathcal{O}_{(3)}(X_i). \tag{2.7}$$

By consistency with the coassociativity condition (2.4) for $k = 2$, $\Delta_{(2)}(X_i)$ contributions must satisfy a set of well-precise conditions. A contribution in $z^2$ is determined by $\delta$, while the other possible contributions are consistent with zero because they are proportional to arbitrary parameters which are independent of $\delta$, as described in section 6. As the analytical procedure requires us to include only the changes imposed by $\delta$, all these last contributions are put to zero. Thus $\Delta_{(2)}(X_i)$ is obtained and found proportional to $z^2$. As equation (2.7) can be easily generalized to

$$\mathcal{O}_{(m)}(X_i) = \Delta_{(m)}(X_i) + \mathcal{O}_{(m+1)}(X_i), \tag{2.8}$$

we have now

$$\Delta(X_i) = \Delta_{(0)}(X_i) + z\delta(X_i) + \Delta_{(2)}(X_i) + \Delta_{(3)}(X_i) + \mathcal{O}_{(4)}(X_i), \tag{2.9}$$

where $\Delta_{(2)}(X_i)$ is known and $\Delta_{(3)}(X_i)$ must be found solving equation (2.4) for $k = 3$. After a new elimination of unwanted contributions, a $z^3$-proportional $\Delta_{(3)}(X_i)$ is thus obtained.

The procedure can thus be iterated obtaining—after that all $\Delta_{(l)}(X_i)$ for $l < m$ have been obtained in the same way—$\Delta_{(m)}(X_i)$ that is found to be a polynomial of degree $m+1$ in $X_j \otimes 1$ and $1 \otimes X_j$ and proportional to $z^m$.

Once all the $\Delta_{(m)}$ are known, the order-by-order commutation relations are obtained from the homomorphism relation (2.5) and, finally, the full coproducts and commutation relations are obtained as formal series.

## 3. Standard quantization of $su(2)$

To enlighten the details of the construction we discuss explicitly the standard deformation of $su(2)$. The standard $(su(2), \delta)$ bialgebra, in the Cartan basis $\{H, X, Y\}$, is given by the cocommutator map ($\delta : g \to g \otimes g$)

$$\delta(H) = 0, \qquad \delta(X) = H \wedge X, \qquad \delta(Y) = H \wedge Y,$$

and the commutation rules

$$[H, X] = X, \qquad [H, Y] = -Y, \qquad [X, Y] = 2H. \tag{3.1}$$

As stated in section 2, we begin to search of the coproducts, starting from the Lie coalgebra $\Delta_{(0)}$ and finding the $\Delta_{(k)}$ imposed by $\delta$ to be consistent with equation (2.4).

The case of $H$ is simple: we start with $\Delta_{(0)}(H) = H \otimes 1 + 1 \otimes H$ and $\delta(H) = 0$, which implies that the anti-cocommutative part of $\Delta_{(1)}(H)$ is zero. Equation (2.4) for $k = 1$ (see section 5 for details) gives the solution

$$\begin{aligned}\Delta_{(1)}(H) = {} & \alpha_1 H \otimes H + \alpha_2(H \otimes X + X \otimes H) + \alpha_3(H \otimes Y + Y \otimes H) \\ & + \alpha_4 X \otimes X + \alpha_5(X \otimes Y + Y \otimes X) + \alpha_6 Y \otimes Y.\end{aligned} \tag{3.2}$$

Since these cocommutative contributions are not related to $\delta(H)$ (that in this case vanishes) and the Hopf algebra axioms are fulfilled whatever the $\alpha_i$ coefficients are, the analytical approach implies that all $\alpha_i = 0$. Thus, in agreement with formula (2.6), we write

$$\Delta(H) = \Delta_{(0)}(H) + \mathcal{O}_{(2)}(H),$$

or, from equation (2.8),

$$\Delta(H) = \Delta_{(0)}(H) + \Delta_{(2)}(H) + \mathcal{O}_{(3)}(H),$$

and as equation (2.4) for $k = 2$ is also consistent with $\Delta_{(2)}(H) = 0$ we have

$$\Delta(H) = \Delta_{(0)}(H) + \Delta_{(3)}(H) + \mathcal{O}_{(4)}(H),$$

where the procedure can be repeated. Thus, for all the orders, the analytical prescription imposes $\Delta_{(k)}(H) = 0, \forall k > 0$. Hence

$$\Delta(H) = \Delta_{(0)}(H) = H \otimes 1 + 1 \otimes H, \tag{3.3}$$

i.e. to a null $\delta$, the analytical procedure associates an object with primitive coproduct. Note that formula (3.3) is not, like in [1], a possible choice but the only coproduct consistent with the analytical prescription.

Equivalently, for $\Delta(X)$ we have

$$\Delta(X) = \Delta_{(0)}(X) + \Delta_{(1)}(X) + \mathcal{O}_{(2)}(X).$$

The coassociativity condition (2.4) for $k = 1$ gives

$$\begin{aligned}\Delta_{(1)}(X) = {} & z\delta(X) + \beta_1 H \otimes H + \beta_2(H \otimes X + X \otimes H) + \beta_3(H \otimes Y + Y \otimes H) \\ & + \beta_4 X \otimes X + \beta_5(X \otimes Y + Y \otimes X) + \beta_6 Y \otimes Y\end{aligned} \tag{3.4}$$

where $\beta_i$ are arbitrary constants which are by no means related to $\delta$. As discussed before (see also section 6), we put $\beta_i = 0$ and have

$$\Delta(X) = \Delta_{(0)}(X) + z\delta(X) + \Delta_{(2)}(X_i) + \mathcal{O}_{(3)}(X).$$

The coassociativity condition (2.4) for $k = 2$ solved in the unknown $\Delta_{(2)}(X)$ gives (again disregarding arbitrary cocommutative contributions independent of $\delta$)

$$\Delta_{(2)}(X) = \frac{z^2}{2}(H^2 \otimes X + X \otimes H^2). \tag{3.5}$$

By repeating this machinery, we write

$$\Delta(X) = \Delta_{(0)}(X) + z\delta(X) + \Delta_{(2)}(X_i) + \Delta_{(3)}(X_i) + \mathcal{O}_{(4)}(X)$$

where now $\Delta_{(2)}(X_i)$ is given by equation (3.5) and $\Delta_{(3)}$ is the new unknown. The coassociativity condition (2.4) for $k = 3$ gives

$$\Delta_{(3)}(X) = \frac{z^3}{6}(H^3 \otimes X - X \otimes H^3) \tag{3.6}$$

and the general formula is obtained by iteration, by neglecting order by order the cocommutative contributions unrelated to $\delta$:

$$\Delta_{(k)}(X) = \frac{z^k}{k!}(H^k \otimes X + (-1)^k X \otimes H^k) \qquad \forall k.$$

Now, $\Delta_{(k)}$ are easily summed to

$$\Delta(X) = \mathrm{e}^{zH} \otimes X + X \otimes \mathrm{e}^{-zH}.$$

The approach is exactly the same for $Y$ and gives a similar result. Thus, we obtain the analytical quantum coproduct associated with $(su(2), \delta)$:

$$\begin{aligned}\Delta(H) &= H \otimes 1 + 1 \otimes H \\ \Delta(X) &= \mathrm{e}^{zH} \otimes X + X \otimes \mathrm{e}^{-zH} \\ \Delta(Y) &= \mathrm{e}^{zH} \otimes Y + Y \otimes \mathrm{e}^{-zH},\end{aligned} \tag{3.7}$$

which, by inspection, are invariant under the combination of flip and $z \to -z$.

Now we have simply to start from the commutators (3.1) and to impose order by order the homomorphism condition for the deformed commutation rules. The quantum commutation rules

$$[H, X] = X, \qquad [H, Y] = -Y, \tag{3.8}$$

are quite easy to find. The remaining one reads

$$[X, Y] = \frac{1}{z}\sinh(2zH), \tag{3.9}$$

which is a combined result of equations (3.8) and (2.5). Note that for $k = 0$ equation (3.9) has to give $[X, Y] = 2H$. This fixes the normalization of the basis and forbids other $z$-dependent commutation rules like

$$[X, Y] = \frac{\sinh(2zH)}{\sinh z}. \tag{3.10}$$

Equations (3.7), (3.8) and (3.9) define uniquely the analytical deformation of the Cartan basis of $su(2)$ such that the $q$-generators $H, X, Y$ could be called the $q$-Cartan basis of $su_q(2)$. By inspection, all the symmetries (for example, $\{H, X, Y\} \leftrightarrow \{H, -X, -Y\}$) and the embedding conditions (for instance, $su(2) \supset \mathrm{borel}(H, X) \supset u(1)$) of the bialgebra $(g, \delta)$ are automatically preserved in the quantization $(g_q, \Delta)$.

The results of this section show that, among all possible coproducts, analyticity chooses the only one invariant under the combination of flip and change of sign in $z$ [6]. Moreover, a $q$-Cartan basis is determined by (3.7), (3.8) and (3.9), in contradistinction to the usual commutation rule (3.10).

Note that the coalgebra (3.7) is consistent with other Lie limits as, for instance, $E(2)$ or, after relabelling the generators, with the twisted jordanian deformation $su_h(2)$ [11]. It is only when also the commutators are included in the game that the one-to-one correspondence between the bialgebra and the quantum algebra is obtained.

As a result, we have obtained a Hopf algebra in which the coproduct map is such that $z\Delta$ is a function of $zX_j \otimes 1$ and $1 \otimes zX_j$, and the commutation rules fulfil that $z[X_l, X_m]$ is a function of $zX_j$. This is a general property of the analytical quantization.

## 4. Standard quantization of $u(3)$

In the previous section, we have discussed the simplest case of $su_q(2)$. The procedure described above can be applied to any bialgebra. As a true example, we give now the standard deformation of $u(3)$. For simple Lie algebras the usual description is made in terms of the Cartan subalgebra, simple roots and the $q$-Serre relations without any reference to non-simple roots that remain undefined [2]. This is a problem for applications where simple and non-simple roots play the same role. We start instead from the Weyl–Drinfeld basis of the bialgebra where all roots are well defined [12, 13] and we obtain a complete description of the whole structure for $u_q(3) \equiv su_q(3) \oplus u(1)$, real form of $A_2^q \oplus A_1$. In this basis, the explicit commutation rules are $(i, j, k = 1, 2, 3)$:

$$\begin{aligned}
&[H_i, H_j] = 0,\\
&[H_i, F_{jk}] = (\delta_{ij} - \delta_{ik})F_{jk},\\
&[F_{ij}, F_{kl}] = (\delta_{jk}F_{il} - \delta_{il}F_{kj}) + \delta_{jk}\delta_{il}(H_i - H_j).
\end{aligned}$$

The canonical Lie bialgebra structure is determined by the cocommutator:

$$\begin{aligned}
&\delta(H_i) = 0,\\
&\delta(F_{ij}) = \frac{z}{2}(H_i - H_j) \wedge F_{ij} + z\sum_{k=i+1}^{j-1} F_{ik} \wedge F_{kj} \qquad (i < j),\\
&\delta(F_{ij}) = \frac{z}{2}(H_j - H_i) \wedge F_{ij} - z\sum_{k=j+1}^{i-1} F_{ik} \wedge F_{kj} \qquad (i > j).
\end{aligned} \tag{4.1}$$

We begin with the coalgebra of the Borel subalgebra $b_+ \equiv \{H_1, H_2, H_3, F_{12}, F_{13}, F_{23}\}$. Repeating the procedure of the preceding paragraph (or, simply, remembering the embeddings $su_q(3) \supset su_q(2)$) we get

$$\begin{aligned}
&\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i,\\
&\Delta(F_{12}) = \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{12} + F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2},\\
&\Delta(F_{23}) = \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{23} + F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2}.
\end{aligned} \tag{4.2}$$

The explicit quantization of $\Delta(F_{13})$ from equation (4.1) requires more work. We find

$$\begin{aligned}
\Delta(F_{13}) = {}& \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{13} + F_{13} \otimes \mathrm{e}^{-z(H_1-H_3)/2} + \ 2\sinh\frac{z}{2}\\
&\times (\mathrm{e}^{z(H_2-H_3)/2} F_{12} \otimes \mathrm{e}^{-z(H_1-H_2)/2} F_{23} - \mathrm{e}^{z(H_1-H_2)/2} F_{23} \otimes \mathrm{e}^{-z(H_2-H_3)/2} F_{12}),
\end{aligned} \tag{4.3}$$

which (like 4.1) is inconsistent with the usual definition [14]

$$F'_{13} := \mathrm{e}^{z/2} F_{12} F_{23} - \mathrm{e}^{-z/2} F_{23} F_{12}. \tag{4.4}$$

Cartan matrix, $q$-Serre relations and $q$-commutators do not seem perhaps the simplest approach to quantum algebras. The origin of the definition (4.4) is indeed related to the $q$-Serre relations

$$\begin{aligned}
&F_{12}^2 F_{23} - (\mathrm{e}^z + \mathrm{e}^{-z}) F_{12} F_{23} F_{12} + F_{23} F_{12}^2 = 0,\\
&F_{12} F_{23}^2 - (\mathrm{e}^z + \mathrm{e}^{-z}) F_{23} F_{12} F_{23} + F_{23}^2 F_{12} = 0,
\end{aligned} \tag{4.5}$$

that, using (4.4), can be written as

$$\begin{aligned}
&\mathrm{e}^{z/2} F_{23} F'_{13} - \mathrm{e}^{-z/2} F'_{13} F_{23} = 0,\\
&\mathrm{e}^{z/2} F'_{13} F_{12} - \mathrm{e}^{-z/2} F_{12} F'_{13} = 0,
\end{aligned}$$

showing that $F'_{13}$ $q$-commutes with both $F_{12}$ and $F_{23}$. Anyway, imposing the homomorphism equation (2.3) of the coproducts, we find that the commutators

$$[F_{12}, F_{23}] = F_{13}, \qquad [F_{32}, F_{21}] = F_{31}$$

of the bialgebra remain unchanged in the quantization. In agreement with the quantum theories, where the commutator is connected to the measure, the commutator remains the appropriate map also in the analytical deformation of $g$.

The quantized coproduct of $b_- \equiv \{H_1, H_2, H_3, F_{21}, F_{31}, F_{32}\}$ is similar:

$$\begin{aligned}
&\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i,\\
&\Delta(F_{21}) = \mathrm{e}^{z(H_1-H_2)/2} \otimes F_{21} + F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2},\\
&\Delta(F_{32}) = \mathrm{e}^{z(H_2-H_3)/2} \otimes F_{32} + F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2},\\
&\Delta(F_{31}) = \mathrm{e}^{z(H_1-H_3)/2} \otimes F_{31} + F_{31} \otimes \mathrm{e}^{-z(H_1-H_3)/2}\\
&\qquad + 2\sinh\frac{z}{2}\big(\mathrm{e}^{z(H_2-H_3)/2} F_{21} \otimes \mathrm{e}^{-z(H_1-H_2)/2} F_{32} - \mathrm{e}^{z(H_1-H_2)/2} F_{32} \otimes \mathrm{e}^{-z(H_2-H_3)/2} F_{21}\big).
\end{aligned}$$

All coalgebra is thus known. Now we have to find the deformed commutation rules compatible with the above coalgebra and the $u(3)$ limit.

From $q$-Serre relations (4.5) (as well as from the $\Delta$ isomorphism) we get the only commutation rules for $b_\pm^q$ that are deformed:

$$\begin{aligned}
&[F_{12}, F_{13}] = [F_{12}, [F_{12}, F_{23}]] = 4\left(\sinh\frac{z}{2}\right)^2 F_{12} F_{23} F_{12},\\
&[F_{13}, F_{23}] = [[F_{12}, F_{23}], F_{23}] = 4\left(\sinh\frac{z}{2}\right)^2 F_{23} F_{12} F_{23},\\
&[F_{31}, F_{21}] = [[F_{32}, F_{21}], F_{21}] = 4\left(\sinh\frac{z}{2}\right)^2 F_{21} F_{32} F_{21},\\
&[F_{32}, F_{31}] = [F_{32}, [F_{32}, F_{21}]] = 4\left(\sinh\frac{z}{2}\right)^2 F_{32} F_{21} F_{32}.
\end{aligned}$$

Now we have to consider the crossed commutation relations. We start from $[F_{23}, F_{21}]$. The compatibility with the quantum coproduct leads to the equation

$$\Delta([F_{23}, F_{21}]) = \mathrm{e}^{z(H_1-H_3)} \otimes [F_{23}, F_{21}] + [F_{23}, F_{21}] \otimes \mathrm{e}^{-z(H_1-H_3)}.$$

Thus, in agreement with [2], the unique analytical solutions consistent with the coproduct map for $[F_{23}, F_{21}]$ and $[F_{12}, F_{32}]$ are:

$$[F_{23}, F_{21}] = 0, \qquad [F_{12}, F_{32}] = 0.$$

From the two embedded $su_q(2)$ Hopf subalgebras we get

$$[F_{12}, F_{21}] = \frac{1}{z}\sinh(z(H_1 - H_2)),$$
$$[F_{23}, F_{32}] = \frac{1}{z}\sinh(z(H_2 - H_3)),$$

and, because of the full structure written in terms of the commutation rules, the (not deformed) Jacobi identities can be used as a short cut to derive

$$\begin{aligned}[F_{13}, F_{21}] = [[F_{12}, F_{23}], F_{21}] = -[[F_{21}, F_{12}], F_{23}] &= \frac{1}{z}[\sinh(z(H_1 - H_2)), F_{23}] \\ &= -\frac{2}{z}\sinh\frac{z}{2}\cosh\left(z\left(H_1 - H_2 + \frac{1}{2}\right)\right)F_{23},\end{aligned}$$

and analogously

$$[F_{13}, F_{32}] = \frac{2}{z}\sinh\frac{z}{2}\cosh\left(z\left(H_2 - H_3 + \frac{1}{2}\right)\right)F_{12},$$
$$[F_{12}, F_{31}] = -\frac{2}{z}\sinh\frac{z}{2}\cosh\left(z\left(H_1 - H_2 - \frac{1}{2}\right)\right)F_{32},$$
$$[F_{23}, F_{31}] = \frac{2}{z}\sinh\frac{z}{2}\cosh\left(z\left(H_2 - H_3 - \frac{1}{2}\right)\right)F_{21}.$$

The last relation is computed imposing the homomorphism property, obtaining

$$\begin{aligned}[F_{13}, F_{31}] = \frac{1}{z}\sinh(z(H_1 - H_3)) &+ \frac{2}{z}\left(\sinh\frac{z}{2}\right)^2\sinh(z(H_1 - H_2))\{F_{23}, F_{32}\} \\ &+ \frac{2}{z}\left(\sinh\frac{z}{2}\right)^2\sinh(z(H_2 - H_3))\{F_{12}, F_{21}\}.\end{aligned}$$

## 5. Friedrichs theorem revisited: $U(g) \to (g, \Delta_{(0)})$

The universal enveloping algebra $U(g)$ is defined in terms of an arbitrary set of $n$ basic elements $\{Y_j\}$ on which a PWB basis for the whole $U(g)$ can be built. They are not (in principle) primitive but they are cocommutative.

Here we give a constructive proof of the Friedrichs theorem, building explicitly the primitive generators $\{X_j\}$ in terms of the $\{Y_j\}$. The machinery consists of repeated changes of bases that allow us to obtain each time a better approximation to primitivity where the problem is reformulated at each step in terms of the preceding basis. An infinite iteration of the procedure allows us to find, among the $\infty$-many possible bases in the $U(g)$, the Lie generators.

More explicitly, we consider that $X_i \equiv \lim_{k\to\infty} X_i^k$ where $\left\{X_i^k\right\}$ is a basic set that approximates the Lie–Hopf coproducts up to $\mathcal{O}_{(k)}(X_i)$. The terms $\mathcal{O}_{(m)}(Z_i)$, defined in section 2

as a series of degree greater than $m$ in $Z_j \otimes 1$ and $1 \otimes Z_j$, are, in this section, cocommutative since we are working in $U(g)$.

To begin with, let us define a homogeneous symmetric polynomial of order $m$:

$$P_{(m)}(Z_i) := \sum_{m_i} f_i^{m_1,m_2\cdots m_n} \mathcal{S}_m[(Z_1)^{m_1}(Z_2)^{m_2}\cdots(Z_n)^{m_n}],$$

where the sum on the $m_i$ is restricted to $\sum m_i = m$, $f_i^{m_1,m_2\ldots m_n} \in \mathbb{C}$ and

$$\mathcal{S}_m[(Z_1)^{m_1}(Z_2)^{m_2}\cdots(Z_n)^{m_n}] \equiv \sum_{\sigma\in\mathbf{S}_m} \sigma[(Z_1)^{m_1}(Z_2)^{m_2}\cdots(Z_n)^{m_n}],$$

$\mathbf{S}_m$ being the group of permutations of order $m$.

Now any original basic set $\{Y_i\}$ is a zero approximation to $\{X_i\}$ : $X_i^0 := Y_i$. Indeed

$$\Delta\big(X_i^0\big) = \Delta_{(0)}\big(X_i^0\big) + \mathcal{O}_{(1)}\big(X_i^0\big) = X_i^0 \otimes 1 + 1 \otimes X_i^0 + \mathcal{O}_{(1)}\big(X_i^0\big). \tag{5.1}$$

The explicit form of $\mathcal{O}_{(1)}\big(X_i^0\big)$ in (5.1) is

$$\mathcal{O}_{(1)}\big(X_i^0\big) = \sum c_i^{jl}\left(X_j^0 X_l^0 \otimes 1 + 1 \otimes X_j^0 X_l^0\right) + \sum d_i^{jl} X_j^0 \otimes X_l^0 + \mathcal{O}_{(2)}\big(X_i^0\big)$$

where $c_i^{jl}$ and $d_i^{jl} = d_i^{lj}$ are constants. Again from (2.1) ,

$$\mathcal{O}_{(1)}\big(X_i^0\big) = \Delta_{(1)}\big(X_i^0\big) + \mathcal{O}_{(2)}\big(X_i^0\big), \tag{5.2}$$

and we have to impose on $\Delta_{(1)}\big(X_i^0\big)$ the coassociativity condition (2.4) for $k = 1$ that gives $c_i^{jl} \equiv 0$ , while no more restrictions are found on $d_i^{jl}$. Thus, if we define

$$P_{(2)}\big(X_i^0\big) := \sum d_i^{jl}\big[X_j^0, X_l^0\big]_+ \tag{5.3}$$

we have

$$\Delta_{(1)}\big(X_i^0\big) = \Delta_{(0)}\big(P_{(2)}\big(X_i^0\big)\big) - P_{(2)}\big(X_i^0\big) \otimes 1 - 1 \otimes P_{(2)}\big(X_i^0\big). \tag{5.4}$$

We can thus define the next approximation of the Lie generators

$$X_i^1 := X_i^0 - P_{(2)}\big(X_i^0\big) \tag{5.5}$$

and we get for $\big\{X_i^1\big\}$ a coproduct with *vanishing first-order contributions*:

$$\Delta\big(X_i^1\big) = \Delta_{(0)}\big(X_i^1\big) + \mathcal{O}_{(2)}\big(X_i^0\big).$$

Still more relevant, equation (5.5) allows us to write $\mathcal{O}_{(2)}\big(X_i^0\big)$ in terms of $\big\{X_i^1\big\}$ as $\mathcal{O}_{(2)}\big(X_i^1\big)$:

$$\Delta\big(X_i^1\big) = \Delta_{(0)}\big(X_i^1\big) + \mathcal{O}_{(2)}\big(X_i^1\big).$$

As both the relations (5.2) and (5.4) can be generalized to

$$\mathcal{O}_{(m)}\big(X_i^{m-1}\big) = \Delta_{(m)}\big(X_i^{m-1}\big) + \mathcal{O}_{(m+1)}\big(X_i^{m-1}\big),$$
$$\Delta_{(m)}\big(X_i^{m-1}\big) = \Delta_{(0)}\big(P_{(m+1)}\big(X_i^{m-1}\big)\big) - P_{(m+1)}\big(X_i^{m-1}\big) \otimes 1 - 1 \otimes P_{(m+1)}\big(X_i^{m-1}\big),$$

we are ready for the next step. Imposing the coassociativity property on the most general symmetric polynomial of third order in $X_j^1 \otimes 1$ and $1 \otimes X_j^1$ , we get

$$\mathcal{O}_{(2)}\big(X_i^1\big) = \Delta_{(2)}\big(X_i^1\big) + \mathcal{O}_{(3)}\big(X_i^1\big), \tag{5.6}$$

$$\Delta_{(2)}\big(X_i^1\big) = \Delta_{(0)}\big(P_{(3)}\big(X_i^1\big)\big) - P_{(3)}\big(X_i^1\big) \otimes 1 - 1 \otimes P_{(3)}\big(X_i^1\big). \tag{5.7}$$

With a new change of basis $X_i^2 := X_i^1 - P_{(3)}\big(X_i^1\big)$ we obtain the coproduct of the second approximation $\big\{X_i^2\big\}$ to the generators in terms of the same $\big\{X_i^2\big\}$:

$$\Delta\big(X_i^2\big) = \Delta_{(0)}\big(X_i^2\big) + \mathcal{O}_{(3)}\big(X_i^2\big),$$

now *free from both first- and second-order contributions*.

The procedure can now be easily iterated and the $\Delta_m\left(X_i^{m-1}\right)$ contribution eliminated through a new change of basis that affects the higher orders only. The residual term becomes $O_{(m+1)}\left(X_i^m\right)$ and we get the $m$-order approximation to the Lie generators

$$\Delta\left(X_i^m\right) = \Delta_{(0)}\left(X_i^m\right) + \mathcal{O}_{(m+1)}\left(X_i^m\right).$$

The true generators of the Lie algebra $g$ are (formally) recovered in the limit

$$X_i := \lim_{m\to\infty} X_i^m$$

and, in agreement with the Friedrichs theorem, their coproduct is the primitive one:

$$\lim_{m\to\infty} \Delta\left(X_i^m\right) = \lim_{m\to\infty} \Delta_{(0)}\left(X_i^m\right) = \Delta_{(0)}(X_i) = \Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i.$$

Of course, this coproduct is an algebra homomorphism with respect to the (linear) Lie algebra commutation rules,

$$\Delta[X_i, X_j] = [X_i, X_j] \otimes 1 + 1 \otimes [X_i, X_j]$$

and the $n$ Lie algebra generators are univocally identified in a constructive manner within $U(g)$ pushing away order by order the corrections to a primitive coproduct.

Let us again stress that the central point of this analytical approach to Friedrichs theorem (as well as to its following extension to $U_q(g)$) is that at each order all relations can be rewritten in terms of the corresponding approximations of the generators.

## 6. Extension of Friedrichs theorem: $U_q(g) \to (g_q, \Delta)$

Of course, the Friedrichs theorem is a well-known result. However, it has been described here because the procedure that allows us to individuate the quantum algebra generators $g_q$, among the $\infty$-many possible bases of the $U_q(g)$ is exactly the same that allows us to individuate the generators $g$ among the $\infty$-many possible bases of the $U(g)$: therefore, the analytical approach can be considered as an extension to quantum algebras of the Friedrichs theorem.

The preceding construction indeed works also in the $\delta \neq 0$ case, thus providing us with the prescription for the construction of the almost primitive generators—obtained in section 2 from $(g, \Delta_{(0)})$—starting from an arbitrary set of basic elements of any $U_q(g)$.

As before, we start with an arbitrary set of basic elements $\{Y_j\}$ that define the $U_q(g)$ (and no more a $U(g)$) with, as a classical limit, a Lie bialgebra with $\delta \neq 0$.

Equations (5.1) and (5.2) are still valid but now to the $\Delta_{(1)}$ of (5.4) we have to add the contribution of $\delta$. We have thus

$$\Delta\left(X_i^0\right) = \Delta_{(0)}\left(X_i^0\right) + z\delta\left(X_i^0\right) + \sum d_i^{jl} X_j^0 \otimes X_l^0 + \mathcal{O}_{(2)}\left(X_i^0\right).$$

The same $P_{(2)}\left(X_i^0\right)$ of equation (5.3) allows us to define again $X_i^1 := X_i^0 - P_{(2)}\left(X_i^0\right)$. As this change of variables does effect the $\delta$ contribution only to higher orders, the differences can be included in $\mathcal{O}_{(2)}\left(X_i^0\right)$ (or, equivalently, $\mathcal{O}_{(2)}\left(X_i^1\right)$). We can thus write

$$\Delta\left(X_i^1\right) = \Delta_{(0)}\left(X_i^1\right) + z\delta\left(X_i^1\right) + \mathcal{O}_{(2)}\left(X_i^1\right).$$

The next step is, as in equation (5.6), to introduce $\Delta_{(2)}\left(X_i^1\right)$ :

$$\mathcal{O}_{(2)}\left(X_i^1\right) = \Delta_{(2)}\left(X_i^1\right) + \mathcal{O}_{(3)}\left(X_i^1\right).$$

As in the $\Delta_{(1)}$ case, $\Delta_{(2)}$ has two contributions: one of them proportional to $z^2$ (and imposed by the consistency between $\delta$ and coassociativity) and the other one described in (5.7). Again, the latter can be removed by another change of basis that does not modify the form of the $z$-dependent contributions as the induced modifications can be included in $\mathcal{O}_{(3)}\left(X_i^2\right)$.

This procedure can be iterated. Once the problem is solved for $\Delta_{(m-1)}$, a contribution to $\Delta_{(m)}$, proportional to $z^m$ (cocommutative for $m$ even and anti-cocommutative for $m$ odd), is found while the unessential $z$-independent terms are removed exactly as in the case $\delta = 0$, with a change of basis that does not affect the form of known $z$-depending terms because the introduced changes are always to orders higher of $z^m$ and thus pushed out in $\mathcal{O}_{(m+1)}$. For $m \to \infty$ the same almost primitive coproducts derived from $(g, \Delta_{(0)})$ in section 2 are found to be a basic set for $U_q(g)$. The homomorphism condition imposes, of course, the same deformed commutation rules and we have thus closed the other side of the diagram, finding that $U_q(g)$ has as one of its basic sets the same almost primitive set $(g_q, \Delta)$ that we have obtained before by analytical continuation of the Lie generators.

Note that the deformation does not affect the fundamental trick of the game: the iterative procedure where each order is stated in terms of the preceding ones.

As we have explicitly constructed the transformation between any arbitrary basic set $\{Y_j\}$ and the quantum algebra generators $\{X_j\}$, we have demonstrated the one-to-one correspondence between $U_q(g)$ and $(g_q, \Delta)$. Since in section 2 it has been shown that $(g_q, \Delta)$ is in one-to-one correspondence with $(g, \delta)$, the classification of quantum groups has been reduced to the classification of Lie bialgebras.

## 7. Concluding remarks

The main result of this paper is the construction of the unique almost primitive basic set characterizing the quantum universal enveloping algebra $U_q(g)$, in perfect analogy with the unique primitive basic set determining $U(g)$. As the last one is the Lie–Hopf algebra $(g, \Delta_{(0)})$, we call the first one *quantum algebra* $(g_q, \Delta)$. Hence, a deformed structure that has exactly the same dimensions as the underlying Lie algebra is introduced instead of the $\infty$-dimensional quantum universal enveloping algebra. This quantum algebra could be the essential object to be connected with physical operators, as in the Lie case where the generators do have a precise meaning in terms of symmetry transformations.

We have also shown that the connections between bialgebras, quantum algebras and quantum universal enveloping algebras are always one-to-one, such that the classification problem (as well as equivalence relations and embedding properties) can be stated at the Lie bialgebra level. As a third point, note also that the analytical quantization method here presented is constructive and could be implemented by making use of computer algebra.

Besides the proposed almost primitive basis, we would like to quote two other relevant bases that play a role both in mathematics and in physics: the Lie basis and the canonical/crystal basis. In the Lie basis (for instance, in twisted deformations) the algebra remains unmodified and the quantization affects only the coalgebra thus offering a possible way to introduce an interaction but saving the global symmetry [15]. Instead, in the canonical or crystal basis (with applications in statistical mechanics [16] and in genetics [17])) the algebraic sector of the Hopf algebra structure is obtained in the limit $|z| \to \infty$ [18, 19] and the comultiplication map is given as a byproduct.

We would also like to recall previous works on the search for isolating a finite-dimensional set of basic elements inside a quantum universal enveloping algebra. In particular, several classes of the so-called quantum (braided) Lie algebras have been introduced (see [20–23] and references therein). In contradistinction to our approach, all these algebras are based on different generalizations of the commutator bracket, either by making use of the adjoint action induced by the Hopf algebra structure or by using $R$-matrix techniques.

Indeed, the fundamental object for our construction of the quantum algebra is the coproduct, while the deformed commutation rules (always given in terms of antisymmetric

commutators) are derived *a posteriori* by making use of the homomorphism property. It is also worth noting that in the usual quantization of simple Lie algebras—as the whole structure is defined in terms of the Cartan subalgebra and simple roots—the $q$-generators associated with non-simple roots (that are as relevant as the others in physics) do not play any role. As a consequence, these $q$-generators can be defined in many different ways, in contradistinction to the Lie case. However the analyticity forbids the $q$-commutator as the appropriate bracket in $U_q(g)$, and non-simple root generators are exactly defined as in the Lie case. Moreover, the fact that the quantum algebra is built up in terms of commutators makes possible a straightforward semiclassical limit in terms of Poisson–Lie structures. In particular, applications of the Poisson $su_q(3)$ algebra will be presented elsewhere.

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## References

[1] Chari V and Pressley A 1994 *A Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[2] Fuchs J 1992 *Affine Lie Algebras and Quantum Groups* (Cambridge: Cambridge University Press)
[3] Lyakhovsky V and Mudrov A I 1992 *J. Phys. A: Math. Gen.* **25** L1139
[4] Mudrov A I 1997 *J. Math. Phys.* **38** 476
[5] Schnider S and Sternberg S 1993 *Quantum Groups from Coalgebras to Drinfeld Algebras* (Cambridge, MA: International Press)
[6] Ballesteros A, Celeghini E and del Olmo M A 2004 *J. Phys. A: Math. Gen.* **37** 4231
[7] Ballesteros A, Celeghini E and del Olmo M A 2005 *J. Phys. A: Math. Gen.* **38** 3909
[8] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)
[9] Jacobson N 1979 *Lie Algebras* (New York: Dover)
[10] Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Lecture Notes in Mathematics* vol 1510, ed P P Kulish (Berlin: Springer) p 221
[11] Ohn Ch 1992 *Lett. Math. Phys.* **25** 85
[12] Ballesteros A, Celeghini E and del Olmo M A 2006 *J. Phys. A: Math. Gen.* **39** 9161
[13] Ballesteros A, Celeghini E and del Olmo M A 2007 *J. Phys. A: Math. Theor.* **40** 2013
[14] Alisauskas S and Smirnov Yu 1994 *J. Phys. A: Math. Gen.* **27** 5925
Alisauskas S and Smirnov Yu 1995 *J. Phys. A: Math. Gen.* **28** 985
[15] Celeghini E and del Olmo M A 2003 *Europhys. Lett.* **61** 438
[16] Kang S J, Kashiwara M, Misra K C, Miwa T, Nakashima T and Nakayashiki A 1992 *Duke Math. J.* **68** 499
[17] Minichini C and Sciarrino A 2006 *Biosystems* **84** 191
[18] Lusztig G 1990 *J. Am. Math. Soc.* **3** 447
Lusztig G 1990 *Prog. Theor. Phys. Suppl.* **102** 175
[19] Kashiwara M 1990 *Commun. Math. Phys.* **133** 249
Kashiwara M 1993 *Duke Math. J.* **69** 455
[20] Delius G W and Huffmann A 1996 *J. Phys. A: Math. Gen.* **29** 1703
[21] Delius G W, Gardner C and Gould M D 1998 *J. Phys. A: Math. Gen.* **31** 1995
[22] Lyubashenko V and Sudbery A 1998 *J. Math. Phys.* **39** 3487
[23] Gurevich D I and Saponov P A 2007 *Preprint* 0705.4167